Feigenvalues for Mandelsets

1991 J. Phys. A: Math. Gen. 24 3363

(http://iopscience.iop.org/0305-4470/24/14/023)

J. Phys. A: Math. Gen. **24** (1991) 3363–3368. Printed in the UK

Feigenvalues for Mandelsets

K M Briggs†§, G R W Quispel† and C J Thompson‡

† Department of Mathematics, La Trobe University, Bundoora, Victoria, Australia 3083
‡ Department of Mathematics, Melbourne University, Parkville, Victoria, Australia 3052

Received 18 March 1991

Abstract. We discuss generalizations of Feigenbaum's constants α and δ to complex polynomial maps of degree higher than 2, and present some numerical estimates. Universality classes are found to depend on the nature of the critical points of the polynomial.

1. Introduction

Since the original discovery by Feigenbaum [1] of universal features of period doubling in one-dimensional maps there has been considerable interest in generalizing these concepts [2] as well as experimental observations of the so-called Feigenbaum constants or Feigenvalues δ and α which characterize the rate of parameter dependent period doubling.

As an example, the map

$$x' = \lambda - |x|^d \qquad \lambda, x, d \in \mathbb{R} \tag{1}$$

has a cascade of birfurcations to 2^k-cycles at parameter values λ_k, $k = 1, 2, \ldots,$ which converge exponentially fast to λ_∞ at a universal rate

$$\delta(d) = \lim_{k\to\infty} (\lambda_{k-1} - \lambda_k)(\lambda_k - \lambda_{k+1})^{-1} \tag{2}$$

with the Feigenvalue $\delta(d)$ depending on the order d of the critical point $x = 0$. For the generic quadratic map one has $\delta(2) = 4.669\,2016\ldots$, for the quartic map one has $\delta(4) = 7.2847\ldots$ and so forth.

The d-dependence of δ and corresponding universality classes of unimodal maps have been studied in some detail [3–12] and it is known, modulo power-law conjugacy [6], that Feigenvalues depend solely on the exponent d of the (single) critical point of the map.

In higher-dimensional maps the universality question is less well understood. Certain classes of constant Jacobian (quadratic) Hénon-type maps are known to undergo period doubling with Feigenvalues [3, 13, 14]

$$\delta = \begin{cases} 4.669\ldots & \text{(dissipative)} \\ 8.721\ldots & \text{(area preserving; symmetric reversible).} \end{cases} \tag{3}$$

§ Present address: Department of Mathematics, Melbourne University, Parkville, Victoria, Australia 3052. Email: kbriggs@mundoe.maths.mu.oz.au

Additionally, it has been claimed [15] on the basis of approximate renormalization arguments that Feigenvalues for the generalized Hénon map

$$x' = 1 - a|x|^d - y \qquad y' = bx \tag{4}$$

depend on the exponent d, and in particular that δ is equal to the one-dimensional $\delta(d)$ for all (dissipative) $|b| < 1$. This claim has no numerical support and it seems in fact, from the available numerical evidence [16], that the Feigenvalues for the map (4) are independent of d and take the values given in (3). The possibility of exponent-dependent Feigenvalues for constant (non-zero) Jacobian maps thus remains an interesting open question.

Complex analytic and area-preserving maps are known to have extended universality classes and corresponding Feigenvalues for general n-tupling (where $n = 2$ corresponds to period doubling etc). Cvitanović and Myrheim [17], for example, studied n-tupling of the complex form

$$z' = \lambda - z^2 \qquad \lambda, z \in \mathbb{C} \tag{5}$$

of the quadratic map and found Feigenvalues $\delta_{m/n}$ for n-tupling with winding numbers m/n $(m = 0, \ldots, n-1)$ corresponding to eigenvalues crossing the unit circle at $\exp(2\pi i m/n)$. We denote this situation by n-tupling(m), omitting the m when it is clear from the context.

Here we report on a preliminary study of universality properties of higher degree polynomial mappings of the complex plane.

2. Complex analytic maps

We begin by considering the elementary polynomial maps of integer degree d:

$$z' = \lambda - z^d \equiv Q_{\lambda,d}(z) \qquad \lambda, z \in \mathbb{C} \tag{6}$$

which have a unique critical point (where the derivative of the map vanishes) at the origin. We are particularly interested in the complex parameter values λ_k corresponding to superstable n^k-cycles (that is, cycles containing the critical point) and the Feigenvalue

$$\delta_{m/n}(d) = \lim_{k \to \infty} (\lambda_{k-1} - \lambda_k)(\lambda_k - \lambda_{k+1})^{-1} \tag{7}$$

corresponding to n-tupling(m).

We now define the *Mandelset* $M(f, c)$ to be the subset of the λ-plane for which the orbit (under an arbitrary complex polynomial f) of the critical point c of f is bounded. Thus a polynomial has as many Mandelsets as critical points. This generalizes the concept of the classical (quadratic) Mandelbrot set.

This definition should be contrasted to that of the *connectivity locus* defined in [19] for cubic maps, where it is a subset of $\mathbb{C}^2$. We find this concept too general for our present purposes.

The λ sequences above thus lie in the Mandelset. The limit points λ_∞ of such sequences can also be expected to lie on the boundary of the corresponding Mandelset with intermediate parameter values for successive n-tupling located at the points of contact of (asymptotically) self-similar components of the Mandelbrot set. As an example the Mandelset $M(\lambda - z^4, 0)$ for the elementary quartic map is shown in figure 1. Note the threefold rotational symmetry. It is a trivial exercise to show that the Mandelset $M(\lambda - z^d, 0)$ has $(d-1)$-fold rotational symmetry.

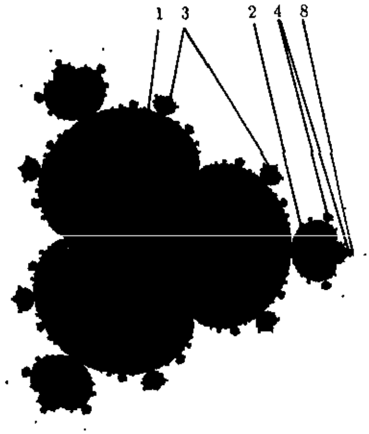

Figure 1. The Mandelset $M(\lambda - z^4, 0)$ of the elementary quartic polynomial. The integer label gives the period of some of the hyperbolic components.

Some numerical Feigenvalues $\delta_{m/n}(d)$ and $\alpha_{m/n}(d)$ for n-tupling(m) of (6) are given in table 1 with $\delta_{m/n}(d)$ defined by (7) and, for a map f_λ with critical point c of order d,

$$\alpha_{m/n}(d) = \lim_{k\to\infty} [c - f_{\lambda_k}^{[n^k]}(c)][c - f_{\lambda_{k+1}}^{[n^{k+1}]}(c)]^{-1} \tag{8}$$

where $f_\lambda^{[N]}$ denotes f_λ composed with itself N times and λ_k is the parameter value corresponding to the superstable n^k-cycle.

More complicated polynomial maps with more than one critical point can also be studied by the above methods. For example the map

$$z' = \lambda + z^4(20z^2 - 48z + 30) \tag{9}$$

is easily shown to have two critical points: at $z = 0$ with $d = 4$ and at $z = 1$ with $d = 3$. The computed Feigenvalues $\delta_{m/n}(d)$, $\alpha_{m/n}(4)$ are given in table 1. Notice the agreement with the elementary $d = 3$ and $d = 4$ values.

Table 1. Feigenvalues α and δ computed directly from equations (7) and (8). Here c is the critical point, d its degree and n the tupling value. For $n = 2$, $m = 1$; and $n = 3$, $m = 1$ or 2.

Function	d	c	n	α	δ
$\lambda - z^2$	2	0	2	-2.502	4.6692
		0	3	$-2.0969 - 2.35828i$	$4.600 + 8.981i$
$\lambda - z^3$	3	0	2	$-0.2518 + 1.8647i$	$3.031 - 4.556i$
		0	3	$-1.3812 - 2.0149i$	$15.795 + 0.5758i$
$\lambda - z^4$	4	0	2	-1.6093	7.2847
		0	3	$-1.9768 - 0.6356i$	$13.035 + 17.795i$
$\lambda - z^5$	5	0	2	$-1.157 + 1.099i$	$7.851 - 5.347i$
$\lambda + z^4(20z^2 - 48z + 30)$	3	1	2	$-0.252 + 1.86i$	$3.031 - 4.456i$
			3	$-1.3812 - 2.0147i$	$15.795 + 0.5758i$
	4	0	2	-1.69	7.2847
			3	$-1.97 - 0.635i$	$13.035 + 17.795i$

Also the Mandelbrot sets for (9) obtained by iterating from $z=0$ and $z=1$ appear to be asymptotically similar to the Mandelbrot sets obtained by iterating (6) from the origin with $d=4$ and 3 respectively. Further work is in progress to see if this phenomenon is generic.

3. Universal equations for n-tupling

The renormalization group method [1] can also be applied to complex maps [17] with universality classes of functions corresponding to n-tupling obtained from solutions of the functional equation

$$g(z)=\alpha g^{[n]}(z/\alpha) \tag{10}$$

where $g^{[n]}$ denotes g composed with itself n times and $g(0)=1$. For n-tupling(m)

$$\alpha=\alpha_{m/n}=[g^{[n-1]}(1)]^{-1} \tag{11}$$

where $\alpha_{m/n}$ is defined by (8) and the corresponding $\delta_{m/n}$ can be computed from g by an asymptotic functional iteration process [6] or from a related universal equation [10, 17].

Solutions of (10) with

$$g(z)=1+g_1z^d+g_2z^{2d}+\cdots \tag{12}$$

yield Feigenvalues for universality classes of functions having $z=0$ as a critical point with exponent d. Direct substitution of (12) into (10) provides one method of solution. An alternative successive approximation method, due to van der Weele *et al* [10], begins with a zeroth-order approximation of the form $g(z)=1+g_1z^d$ and yields rapid convergence for α in particular. This method exploits the representation of the solution g as

$$g(z)=\lim_{k\to\infty} g_k(z) \qquad g_k(z)=\mu\alpha^k f_{\lambda_\infty}^{[n^k]}(z/(\mu\alpha^k)) \tag{13}$$

(where $f_\lambda(z)\equiv 1-\lambda z^d$). This, together with the relations (for $n=2$)

$$g(0)=1 \qquad g(1)=\alpha^{-1} \qquad g'(1)=\alpha^{d-1} \tag{14}$$

suffices to determine μ α, and λ_∞ and hence $g(z)$ for period doubling. We have solved equations (14) by using a Newton–Raphson iteration in three-dimensional complex space to find μ, α, and λ_∞ in the case $n=2$. It was found necessary to use multiple-precision arithmetic, principally because a large number of iterations of f (n^k, k typically up to 10) are involved, in which round-off error must be kept under control. With intermediate calculations to about 50 decimal places, typically 4 or 5 correct decimal places are obtained in the final α. Note the agreement between tables 1 and 2.

δ is more difficult to compute. We used two methods: that of van der Weele *et al* [10] and that of McGuire and Thompson [6]. The latter was usually more rapidly convergent. Table 2 gives a synthesis of the two methods, based on our experience of the numerical behaviour of these methods. The results, although inaccurate, are sufficient to confirm agreement with table 1.

One interesting feature of this method is that for fixed $d\geqslant 4$ and $n=2$ one obtained multiple complex solutions. Some typical results are given in table 2. In this particular case it appears that the number of distinct solutions of (10) (with $n=2$) of the form

Table 2. Feigenvalues α and δ computed from the functional equation (10) with $n=2$. * denotes very uncertain quantities. Here d is the order of the critical point of the map f. The last column examines a relation suggested by reference [8].

d	α	δ	$(2d-1)\delta/(2d-2)\alpha^d$
2	−2.502 907 875	4.669	1.118
3	−0.251 81 ± 1.864 695i	3.303 ∓ 4.45i	1.01 ± 0.25i
4	−1.690 302	7.28	1.06
	0.605 56 ± 1.509 46i	*2 ∓ 5.7i	*0.97 ± 0.14i
5	0.8978 ± 1.1812i	*1.3 ∓ 6.3i	*0.51 ± 1.5i
	−1.1568 ± 1.09914i	7.85 ± 5.34i	1.03 ± 0.06i
6	−1.4677	9.3	1.09
	1.015 ± 0.9536i	*0.9 ∓ 6.4i	*0.0042 ± 0.056i
	−0.5188 ± 1.4061i	6.3 ∓ 8.4i	1.07 ± 0.06i
7	1.067 ± 0.794i	*0.7 ∓ 6.5i	*0.74 ± 0.35i
	−1.12537 ± 0.7232i	11.1 ∓ 6.0i	1.17 ± 0.06i
	−0.0699 ± 1.4338i	5.0 ∓ 10.7i	1.013 ± 0.1i
8	−1.35798	11	1.02
	1.091 ± 0.677i	*4.7 ∓ 6.6i	*0.68 ± 0.89i
	0.23619 ± 1.3633i	4.0 ∓ 12i	1.02 ± 0.08i
	−0.8950 ± 1.0766i	10.086 ∓ 8.98i	1.02 ± 0.04i
9	1.102 ± 0.5897i	*0.4 ∓ 6.6i	*0.90 ± 0.32i
	0.444 ± 1.2668i	3.0 ∓ 1.3i	1.06 ± 0.04i
	−0.5445 ± 1.2416i	8.7 ∓ 1.19i	1.04 ± 0.03i
	−1.250 ± 0.5314i	13.5 ∓ 6.77i	1.04 ± 0.01i

(12) is the integral part of $d/2$. It remains however, to determine which of these solutions correspond to bona fide n-tupling along continuous paths through the appropriate Mandelset. We can confirm for the case $d=4$, 5 and 6, that the points λ_∞ appear graphically to lie on the boundary of the Mandelset.

Asymptotic forms of feigenvalues for large n [17] and large d suggested by Delbourgo and Kenny [8] and the possible role of conjugacy classes in classifying universality classes of functions with multiple critical points are among the many problems currently being investigated. In particular, the asymptotic relation

$$\lim_{n\to\infty} \frac{(2d-1)\delta_{m/n}}{(2d-2)\alpha^d_{m/n}} = 1 \tag{15}$$

suggested in [8] for real maps appear to be quite accurate in the complex case even for $n=2$ (see the last column of table 2).

Real maps in two or more dimensions pose similar universality questions for n-tupling. See also reference [18] for an extension to non-analytic complex maps.

Acknowledgments

The multiple-precision computations described above were performed with the software package MPFUN, written by David Bailey of NASA Ames Research Center. We would like to thank Mitchell Feigenbaum for useful discussions.

References

[1] Feigenbaum, M J 1979 *J. Stat. Phys.* **21** 669
[2] Cvitanović, P (ed) 1984 *Universality in Chaos* (Bristol: Adam Hilger)
[3] Derrida B, Gervois A and Pomeau Y 1979 *J. Phys. A: Math. Gen.* **12** 269
[4] Collet, P, Eckmann J-P and Lanford II O E 1980 *Commun. Math. Phys.* **76** 211
[5] Chang S-J, Wortis M and Wright J A 1981 *Phys. Rev.* A **24** 2669
[6] McGuire J B and Thompson C J 1982 *J. Stat. Phys.* **27** 183
[7] Eckmann J-P and Wittwer P 1985 Computer Methods and Borel Summability Applied to Feigenbaum's Equation (*Lecture Notes in Physics* **227**) (Berlin: Springer)
[8] Delbourgo R and Kenny B G 1986 *Phys. Rev.* A **33** 3292
[9] van der Weele J P, Capel H W and Kluiving R 1986 *Physica* **119A** 15
[10] van der Weele J P, Capel H W and Kluiving R 1987 *Physica* **145A** 425
[11] Thompson C J and McGuire J B 1988 *J. Stat. Phys.* **51** 991
[12] Briggs K M 1991 *Mathematics of Computation* in press
[13] Feigenbaum M J 1980 *Los Alamos Science* **1** 4
[14] Greene J M, Mackay R S, Vivaldi F and Feigenbaum M J 1981 *Physica* **3D** 468
[15] Hauser P R, Curado E M F and Tsallis C 1985 *Phys. Lett.* **108A** 308
[16] Roberts J A G unpublished results
[17] Cvitanović P and Myrheim J 1989 *Commun. Math. Phys.* **121** 225
[18] Gunaratne G 1987 *Phys. Rev.* A **36** 1834
[19] Branner B and Hubbard J H 1988 *Acta Math.* **160** 143